MASARYKOVA UNIVERZITA
Přírodovědecká fakulta
Ústav matematiky a statistiky

Diplomová práce

Brno 2025 Štěpán Zapadlo

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Computational analysis in neuroscience

Diplomová práce

Štěpán Zapadlo

Vedoucí práce: doc. RNDr. Lenka Přibylová, Ph.D. **Brno 2025**

Bibliografický záznam

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Název práce:	Computational analysis in neuroscience
Studijní program:	Matematika
Studijní obor:	Modelování a výpočty
Vedoucí práce:	doc. RNDr. Lenka Přibylová, Ph.D.
Akademický rok:	2024/2025
Počet stran:	xv + 91
Klíčová slova:	Práce; Něco; Klíčové slovo

Bibliographic Entry

Author:	Štěpán Zapadlo Faculty of Science, Masaryk University Department of Mathematics and Statistics

Title of Thesis:	Computational analysis in neuroscience
Degree Programme:	Mathematics
Field of Study:	Modelling and computations
Supervisor:	doc. RNDr. Lenka Přibylová, Ph.D.
Academic Year:	2024/2025
Number of Pages:	xv + 91
Keywords:	Thesis; Keyword; Long word

Abstrakt

Abstrakt **to jest**

Abstract

Aaaa

M U N I
S C I

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ZADÁNÍ DIPLOMOVÉ PRÁCE

Akademický rok: 2024/2025

Ústav:	Přírodovědecká fakulta
Student:	Bc. Štěpán Zapadlo
Program:	Aplikovaná matematika
Specializace:	Modelování a výpočty

Ředitel *ústavu* PřF MU Vám ve smyslu Studijního a zkušebního řádu MU určuje diplomovou práci s názvem:

Název práce:	Computational analysis in neuroscience
Název práce anglicky:	Computational analysis in neuroscience
Jazyk závěrečné práce:	angličtina

Oficiální zadání:
The thesis will investigate the potential of computational methods for analyzing neuronal data and models. The aim will be to explore various novel algorithms and techniques related to EEG signals and its modeling (including optimization methods, continuation methods, machine learning and artificial intelligence). Evaluate and compare different computational approaches and algorithms for analyzing neuroscience data, aiming to provide guidelines and best practices for researchers in the field.

Literatura:
KUZNECOV, Jurij Aleksandrovič. *Elements of applied bifurcation theory*. 2nd ed. New York: Springer-Verlag, 1998, xviii, 591. ISBN 0387983821.

BRUNTON, Steven L.; KUTZ, J. Nathan. Data-Driven Science and Enginee- ring: Machine Learning, Dynamical Systems, and Control. Cambridge University Press, 2019. Dostupné z doi: 10.1017/9781108380690

Vedoucí práce:	doc. RNDr. Lenka Přibylová, Ph.D.
Datum zadání práce:	18. 8. 2023
V Brně dne:	24. 9. 2024

Zadání bylo schváleno prostřednictvím IS MU.

Bc. Štěpán Zapadlo, 17. 10. 2023
doc. RNDr. Lenka Přibylová, Ph.D., 18. 10. 2023
RNDr. Jan Vondra, Ph.D., 19. 10. 2023

Poděkování

Chci poděkovat

Prohlášení

Dělal jsem na tom sám a celé stack overflow

Brno 24.09.2024

. . . .
Štěpán Zapadlo

Table of contents

Chapter 1

Introduction

Synchronization is a phenomenon commonly found in nature and, as is often the case, it takes many shapes and forms. From the coordination between neurons in our brains or fire-fly lights, to synchronization found in electrical grids and financial markets. What started in the 17th century with Huygens' investigation into the behavior of two weakly interacting pendulum clocks through a heavy beam (see Willms, Kitanov, and Langford 2017) has since evolved into a field rich with both theory and applications.

The key application and motivation of this thesis lies, as the name may suggest, in brain dynamics. Several studies have shown that high-frequency oscillations (HFOs), very high-frequency oscillations (VHFOs), and even ultra-fast oscillations (UFOs) in electroencephalographic (EEG) recordings measured deep in the brain could be potentially used as biomarkers of epileptogenic zones of focal epilepsy. Furthermore, there is also evidence they correlate with the severity of epilepsy. Research suggests that higher frequencies oscillations (VFHOs and UFOs) are more local, i.e. spatially restricted, than traditional HFOs, thus providing better a guidance in locating the areas of epilepsy. Fast oscillations lie outside the realm of physiologically possible frequencies of single neurons. This indicates another mechanism must be at play, but its identification is an open question in neuroscience.

Primary goal of this thesis is to provide an insight into a small part of computational analysis in neuroscience – the phase synchronization[1] of small networks of neurons, behaviors arising from this phenomenon and techniques used in its exploration. Nonetheless, other models and applications will be considered when appropriate. To be more precisely, we will mainly be concerned with different methods of the computation exploration of this phenomenon in various models.

[1]By phase synchronization we mean the state when the difference in phase of the oscillators remains bounded while e.g. their amplitudes may differ (Pikovsky, Rosenblum, and Kurths 2001).

1.1 Approaches

Simulation v. Bifurcations (maybe add later)

1.2 Structure

1. Introduction of the importance of synchronization on (multiple) examples
    1. biomarker for epilepsy
    2. in other fields…
    3. And explanation that we will mainly focus on comparison of simulation vs bifurcations (not anything else, though it is a vast topic)
        - It is NOT a frequency analysis thesis
2. Theoretical introduction to ODEs and DDEs and 1D optimization methods
    1. ODE, DDE, limit cycle
    2. introduce numerical solvers of ODEs (RK45, Euler-Maruyama)
    3. introduce method of steps
    4. introduce basic 1D optimization methods
3. Introduction to neuron models (and mathematical modeling in neuroscience in general)
    1. probably Interneuron, VdP, but possibly others (Hodgkin-Huxley formalism, HH type models)
    2. explain types of coupling
    3. Showcase of NeuronToolbox.jl
        - how it simplifies the code and makes it more readable
        - TODO: improve the usefulness and the code
        - TODO: add support for DDEs
        - TODO: rework all the examples to use this library
        - TODO: publish it :)
4. Simulation approach
    1. first start with the intuitive approach
    2. each time show its "performance" (and time/memory complexity)
    3. introduce different period & shift searching techniques
    4. introduce different starters, indexers and iterators (and really explain the need for all of these concepts)
    5. introduce periodicity checkers
    6. briefly mention multithreading, metacentrum and cloud computing?
5. Bifurcations approach
    1. introduce bifurcation theory with emphasis on DDEs and mainly continuations of periodic orbits
        1. include a treatment of collocations, newtons method, basic continuation description
    2. describe theory and numerical method for computing Floquet multipliers
    3. explain/show implementation for DDEBifurcationKit.jl

        - TODO: Actually do the implementation
    4. show the same example as in Simulation approach computed with bifurcations
6. Compare Simulations vs Bifurcations
    1. compare the time/memory/theory/implementation complexity
    2. compare the parametric dependence
    3. compare the accuracy
7. Conclusion

Chapter 2

A primer on dynamical systems

In the Introduction 1, we have motivated the entire thesis with the usefulness of the knowledge and of the understanding of synchronization in neuroscience. But we will not describe any experiments performed on real couples of neurons in a lab. Instead, we shall deal with the mathematical abstraction for the studied object, e.g. the coupled neurons.

This abstraction is typically called a (mathematical) model (of the reality). The model should, in theory, capture all the important characteristics of the underlying reality. If the state of the model evolves in time, e.g. a model of neuron starts spiking, we usually call this model a dynamical system.

Definition 2.1 (Dynamical system)**.** A *dynamical system* is a triple $\{\mathbb{T}, \mathbb{X}, \varphi^t\}$, where $\mathbb{T} \subseteq \mathbb{R}$ (*time*) endowed with addition $+$ is a subgroup of $(\mathbb{R}, +)$, $\mathbb{X}$ is a metric space (called a *state space*), and $\{\varphi^t\}_{t\in\mathbb{T}}$ is a family of evolution operators parametrized by $t \in \mathbb{T}$, such that $\varphi^t : \mathbb{X} \to \mathbb{X}$ maps a certain point $x_0 \in \mathbb{X}$ to some other state $x_t = \varphi^t x_0 \in \mathbb{X}$.

In the Definition 2.1, the time set $\mathbb{T}$ can take on various forms. In ecology, we often see a discrete $\mathbb{T} = \mathbb{N}_0$ or $\mathbb{T} = \mathbb{Z}$ representing a yearly interval between measurements of our system. On the other hand, in physics (and neuroscience) we typically employ $\mathbb{T} = \mathbb{R}$ as we are concerted with even the shortest time intervals and associated changes. Similarly, the exact choice of the state space $\mathbb{X}$ is dependent of the system in question, but typically we use $\mathbb{R}^n$.

Right now, nothing in the Definition 2.1 guarantees the system does not abruptly change state, because in general $x \neq \varphi^0 x$. If this equality does not hold for at least one $x \in \mathbb{X}$, then such system is called *stochastic*.

Definition 2.2 (Deterministic dynamical system)**.** A dynamical system, see Definition 2.1, is called *deterministic* if and only if the following condition is fulfilled

$$\varphi^0 = \mathrm{id}, \tag{2.1}$$

in other words $\forall x \in \mathbb{X} : x = \varphi^0 x$.

Onwards, we will predominantly use deterministic dynamical systems. Another assumption we shall make throughout this thesis is that the "laws of nature" do not change in time, i.e., we presume the dynamical system in question is autonomous (although it may depend on the past).

Definition 2.3 (Autonomous dynamical system)**.** A deterministic dynamical system, see Definition 2.2, is called *autonomous* if and only if the following condition is fulfilled

$$\forall t, s \in \mathbb{T} : \varphi^{t+s} = \varphi^t \circ \varphi^s, \tag{2.2}$$

in other words $\forall x \in \mathbb{X}\ \forall t, s \in \mathbb{T} : \varphi^{t+s}x = \varphi^t(\varphi^s x)$.

Most often, a dynamical system is given implicitly by some differential equation, be it an ordinary differential equation (ODE), e.g.[1]

$$\dot{x}(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t) \cdot r_0 \cdot \left(1 - \frac{x(t)}{K}\right), \tag{2.3}$$

or a delay differential equation (DDE), for example a modified (2.3)

$$\dot{x}(t) = x(t-\tau) \cdot r_0 \cdot \left(1 - \frac{x(t)}{K}\right),$$

where $\tau > 0$.

2.1 Basic concepts

In this section, we shall introduce basic concepts regarding dynamical systems including, but not limited to, notions of certain special solutions and their stability. Little comment beside the definitions themselves will be provided, as an interested reader can find much more in

Definition 2.4 (Orbit)**.** An *orbit (trajectory)* with an *initial point* $x_0 \in \mathbb{X}$ is an ordered subset of the state space $\mathbb{X}$,

$$Or(x_0) = \left\{x \in \mathbb{X} \mid x = \varphi^t x_0, \forall t \in \mathbb{T} \text{ such that } \varphi^t x_0 \text{ is defined}\right\}$$

In the case of a continuous dynamical system, the orbits are *oriented curves* in the state space. For a discrete dynamical systems, they become sequences of points in $\mathbb{X}$.

Definition 2.5 (Phase portrait)**.** A *phase portrait* of a dynamical system is a partitioning of the state space into trajectories.

Definition 2.6 (Equilibrium)**.** A point $x_0 \in \mathbb{X}$ is called an *equilibrium* (fixed point) if $\varphi^t x_0 = x_0$ for all $t \in \mathbb{T}$.

[1] The equation (2.3) describes the Verhulst model of a population (and its growth, characterized by r_0) in some closed environment with some finite capacity (controlled by K).

Definition 2.7 (Cycle)**.** A *cycle* is a periodic orbit, namely a non-equilibrium orbit L, such that each point $x_0 \in L$ satisfies $\varphi^{t+T} x_0 = \varphi^t x_0$ with some $T > 0$, for all $t \in \mathbb{T}$. The smallest admissible T is called the *period* of the cycle L.

Definition 2.8 (Invariant set)**.** An *invariant set* of a dynamical system $\{\mathbb{T}, \mathbb{X}, \varphi^t\}$ is a subset $\mathbb{S} \subset \mathbb{X}$ which satisfies

$$x \in \mathbb{S} \implies \varphi^t x \in \mathbb{S} \; \forall t \in \mathbb{T}.$$

Definition 2.9 (ω-limit and α-limit point)**.** A point $x_* \in \mathbb{X}$ is called an *ω-limit point* (resp. α-limit point) of the orbit $Or\,(x_0)$ starting at $x_0 \in \mathbb{X}$ if their exists a sequence of times $\{t_k\}_{k=1}^{\infty} \subseteq \mathbb{T}$ with $t_k \to \infty$ (resp. $t_k \to -\infty$), such that

$$\varphi^{t_k} x_0 \underset{k\to\infty}{\longrightarrow} x_*.$$

Definition 2.10 (ω-limit and α-limit set)**.** A set $\Omega(Or\,(x_0))$ of all ω-limit points of the orbit $Or\,(x_0)$, see Definition 2.9, is called an ω-limit set. Similarly, a set $\mathbb{A}(Or\,(x_0))$ of all α-limit points of the orbit $Or\,(x_0)$ is called an α-limit set.

Lastly, a set $\Lambda(Or\,(x_0)) = \Omega(Or\,(x_0)) \cup \mathbb{A}(Or\,(x_0))$ of all limit points of the orbit $Or\,(x_0)$ is called its *limit set*.

Definition 2.11 (Limit cycle)**.** A *limit cycle* is a cycle of a dynamical system, see Definition 2.7, which is also a limit set, see Definition 2.10, of neighboring orbits.

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